MODE COUPLING THEORY

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A mode coupling theory for bunched beam instability, which is similar to Satoh's theory[1], is developed for a Gaussian beam. The theory converts Sacherer's integral equation with mode coupling[2] into a matrix eigenvalue problem. The present theory assumes well-defined azimuthal modes and takes into account radial modes which are expressed as superpositions of orthogonal functions. The method is an extension of the method of Besnier[3] who applied it to solve Sacherer's integral equation without mode coupling.

We start from Sacherer's integral equation with mode coupling[2]

$$\frac{m-\lambda}{m} R_m(r) = i \frac{e\omega_0}{T\Omega_s} \frac{1}{r} \frac{d\psi_0}{dr} \sum_{p=-\infty}^{\infty} \frac{Z(p\omega_0+\Omega)}{p+\Omega/\omega_0}$$

$$\times \; J_m((p+\frac{\Omega}{\omega_0})r) \, i^m \sum_{n=-\infty}^{\infty} i^{-n} \int_0^\infty R_m(r') J_m((p+\frac{\Omega}{\omega_0})r')r'dr' \quad , \tag{1}$$

where $R_m(r)\exp(im\phi)$ is a perturbed distribution function for azimuthal mode m, the variables r and ϕ are defined by

$$\theta = r\cos\phi \quad , \tag{2}$$

$$\frac{k_0}{\Omega_s}\varepsilon = r\sin\phi \quad , \tag{3}$$

and $\psi_0(r)$ is an unperturbed stationary distribution function. The other notations are defined as follows:

$\lambda = \Omega/\Omega_s$
Ω_s: incoherent synchrotron frequency
Ω: coherent frequency
ω_0: revoltuion frequency
e: elementary charge
T: revolution period
$Z(\omega)$: coupling impedance
J_m: Bessel function of order m
θ: longitudinal angular position with respect to the bunch center
ε: energy error
$k = \alpha\omega_0/E$
E: energy
α: momentum compaction factor.

We first note from eq. (1) that

$$\frac{m+\lambda}{m} R_{-m}(r) = \frac{m-\lambda}{m} R_m(r) \quad , \tag{4}$$

and the functional dependence of $R_m(r)$ for positive and negative m is the same except for constant factors. Thus we expand $R_m(r)$ as

$$R_m(r) = W(r) \sum_{k=0}^{\infty} a_k^{(m)} f_k^{(|m|)}(r) \quad , \tag{5}$$

using orthogonal functions $\{f_k^{(|m|)}(r)\}$ which are normalized as

$$\int_0^\infty W(r)\, f_k^{(|m|)}(r)\, f_\ell^{(|m|)}(r)\, r\, dr = \delta_{k\ell} \quad . \tag{6}$$

The weight function W(r) is defined as

$$W(r) = C \frac{1}{r} \frac{d\psi_0}{dr} , \tag{7}$$

where the normalization constant C is chosen as

$$C = - \frac{2\pi \nu_s E \sigma^2}{N e \alpha} . \tag{8}$$

Here ν_s is a synchrotron tune given by Ω_s/ω_0, σ is the rms bunch length divided by the average radius of a storage ring and N is the number of particles in a bunch. For a Gaussian bunch relevant to an electron storage ring, the orthogonal function $f_k^{(|m|)}(r)$ is chosen as

$$f_k^{(|m|)}(r) = \sqrt{\frac{k!}{(|m| + k)!}} \left(\frac{r}{\sqrt{2}\sigma}\right)^{(|m|)} L_k^{(|m|)}\left(\frac{r}{\sqrt{2}\sigma}\right) , \tag{9}$$

where $L_k^{(|m|)}(x)$ is the generalized Laguerre function.

Using the orthogonality relation (6), Sacherer's integral equation (1) is transformed to the following matrix equation

$$\lambda a_h^{(m)} = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \{m\delta_{mn} \delta_{h\ell} - i \frac{e \omega_0 m}{T\Omega_s C} M_{n\ell}^{mh}\} a_\ell^{(n)} , \tag{10}$$

where

$$M_{n\ell}^{mh} = \sum_{p=-\infty}^{\infty} \frac{Z(p\omega_0+\Omega)}{p+\Omega/\omega_0} i^{m-n} I_{mh}(p+\Omega/\omega_0) I_{n\ell}(p+\Omega/\omega_0) , \tag{11}$$

and

$$I_{n\ell}(p') = \int_0^\infty f_\ell^{(|n|)}(r) J_n (p'r) r dr . \tag{12}$$

Using the explicit form of eq. (9), we obtain

$$I_{n\ell}(p') = \frac{1}{\sqrt{(n+\ell)!\ell!}} \left(\frac{\sigma p'}{\sqrt{2}}\right)^{n+2\ell} \exp\left(- \left(\frac{\sigma p'}{\sqrt{2}}\right)^2\right) \quad (n > 0) , \tag{13}$$

$$I_{-n\ell}(p') = (- 1)^n I_{n\ell}(p') . \tag{14}$$

Equation (10) is the matrix equation we have to solve.

The theory is applied to explain the bunch lengthening observed at SPEAR II. Though qualitative explanation can be obtained, there is a discrepancy of several factors. The discrepancy is ascribed to our imperfect knowledge of coupling impedance or to neglected effects such as radiation damping and quantum excitation. The details of this work will be published elsewhere.

References

1) K. Satoh: PEP-Note 361 (1981).
2) Sacherer: IEEE Trans. Nucl. Sci. NS-24, No.3 (1977) 1393.
3) G. Besnier: Nucl. Instrum. & Methods 164 (1979) 235, also thesis Univ. de Rennes (1978).